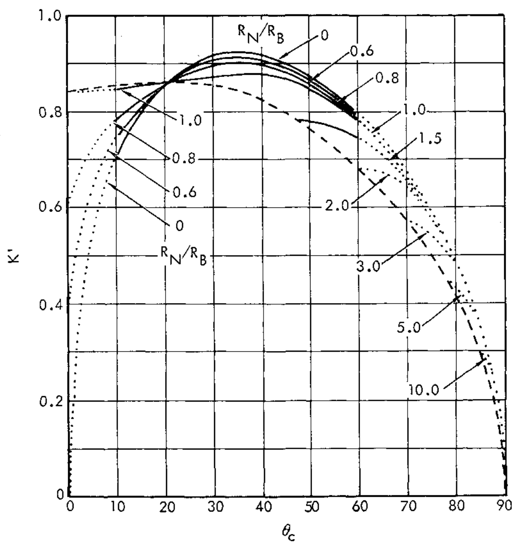

Fig. 3 Averaging coefficient for sphere-cones as a function of cone half-angle.

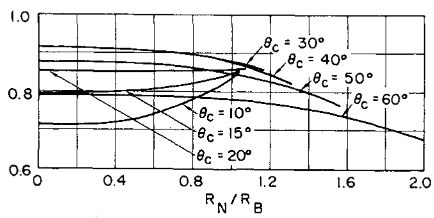

Fig. 4 Averaging coefficient for sphere-cones as a function of base to nose radius ratio.

A computation of K' has been made for a sphere-cone with a cone half-angle of 20° and R_N/R_B of $\frac{1}{3}$ using a more realistic pressure distribution obtained by the method of characteristics. The computed value of K' was 5% less than that computed using Newtonian pressure distribution. For sphere-cones with greater cone half-angles, the difference between the two pressure distributions is less, so that the error in K' as presented also is less.

Summary

The averaging coefficient K for spherical segments varies only from 1.00 to 0.84, and an error of less than 5% in the total heat rate occurs for base to nose radius ratios between zero and 0.85 if a K of 0.95 is used. Thus, for spherical segments,

$$q_t = 0.95\, A_B q_0 \qquad 0 < R_B/R_N < 0.85 \tag{24}$$

is a reasonable equation to use for preliminary design estimates. A slightly more complex equation for total heat rate which is in error by less than 3% and includes all spherical segments through hemispheres is

$$q_t = [1 - 0.128\,(R_B/R_N)^2]\, A_B q_0 \tag{25}$$

For vehicles that are sphere-cones with half-angles between 10° and 60°, the total heat rates may be computed using the equation

$$q_t = 0.91\,(R_N/R_B)^{1/2}\, A_B q_0 \cos^2(\theta_c - 35°) \tag{26}$$

Less than 9% error results from using this simple equation.

Reference

[1] Bromberg, R., Fox, J., and Ackermann, W., "A method of predicting convective heat input to the entry body of a ballistic missile," *Transactions of the First Technical Symposium on Ballistic Missiles: Hyperatmospheric and Hypersonic Phenomena; Properties of Missile Materials* (The Ramo-Wooldridge Corp., Los Angeles, Calif., June 1956), Vol. IV, pp. 159–174.

Analysis of the Flow and Heat Transfer Processes in a Tube Arc for Heating a Gas Stream

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An approximate analysis of the flow and heat transfer processes in a tube arc is presented, the objective of the analysis being to obtain a means for computing the relations between the operating variables, the geometry, and the parameters of the flow exhausting from the device. The analysis employs the approximations usually made for free jet flows and boundary layers. It is assumed that a cool layer of gas wets the wall of the tube. The results, therefore, would be expected to apply for those ranges of the operating variables of a tube arc where the latter condition occurs.

1. Introduction

THIS note discusses methods for analyzing the flow and heat transfer processes in a tube arc, i.e., an electric arc that passes through a narrow cylindrical tube, concurrently with a flow of gas to be heated by the arc discharge. Referring to Fig. 1, the gas to be heated enters the tube at station 1. A portion of the gas is heated by the arc discharge, and the unheated balance of the gas passes near the wall of the tube. The thermal boundary, denoted by δ_t in Fig. 1, encloses the gas heated by the arc discharge.

The following principal assumptions are made in the analysis:

1) The gas is inviscid and in local thermal equilibrium.
2) The flow is steady, laminar, and axisymmetric.
3) The radial velocity component is much smaller than the axial velocity component, and there is no component of velocity in the tangential direction.
4) Ohm's law is valid.
5) Energy transport by radiation is neglected.
6) The Mach number is small.

For clarity, the basic method employed for analyzing the tube arc is first applied herein to an unconfined arc that is subjected to forced convection along its axis.

2. Analysis of an Unconfined Arc in Forced Convection

Figure 2 illustrates schematically the unconfined arc to be discussed. The velocity V of the gas far from the axis of the arc and the pressure gradient dp/dz are given functions of the axial coordinate z. The electric current I, the mass flow rate $\langle m \rangle$, the average rate of energy transport $\langle mh \rangle$, and the average rate of momentum transport $\langle mV \rangle$ through the arc are considered to be known at the axial position z_0. The parameters of the flow and the enthalpy distribution at any position along the arc are to be determined.

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The conservation equations describing the flow and the heat transfer processes of the unconfined arc, in cylindrical coordinates, are as follows:

Continuity

$$\frac{\partial}{\partial r}(r\rho v_r) + \frac{\partial}{\partial z}(r\rho v_z) = 0 \quad (1)$$

where

ρ = the density of the gas
$v_r\ v_z$ = the radial and axial velocity components, respectively

Momentum

$$\rho\left(v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} \quad (2)$$

where

p = the pressure of the gas

Energy

$$\rho\left(v_r\frac{\partial h}{\partial r} + v_z\frac{\partial h}{\partial z}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{rk}{c_p}\frac{\partial h}{\partial r}\right) + \sigma E^2 \quad (3)$$

where

h = the specific enthalpy of the gas
k = the thermal conductivity of the gas
c_p = the specific heat of the gas at constant pressure
σ = the electrical conductivity of the gas
E = the electric field

Ohm's Law

$$I = 2\pi E\int_0^\infty r\sigma\, dr \quad (4)$$

where

I = the arc current

The lateral boundary conditions are

$$\partial h/\partial r = v_r = 0 \text{ at } r = \epsilon$$
$$v_z = V,\ h = h_e \text{ at } r = \infty \quad (5)$$

where

h_e = the specific enthalpy of the gas in the external flow, far from the axis
ϵ = an arbitrarily small radius

The stream function $\psi = g(\xi)f(\eta,\xi)$ and the similarity transformation

$$\eta = \frac{V(z)}{g(z)}\int_0^r r\rho\, dr, \qquad \xi = z \quad (6)$$

(where g is a scale function to be obtained in the solution of the equations) were introduced and applied to the momentum and energy equations. Hence,

$$ff_{\eta\eta} + (gV'/g'V)(\theta/\theta_e - f_\eta^2) = g/g'(f_\eta f_{\eta\xi} - f_\xi f_{\eta\eta}) \quad (7)$$

and

$$g'f H_\eta + \frac{\sigma E^2 g}{\rho V h_e} + \left(\frac{2kH_\eta}{c_p\theta}\int_0^\eta \theta\, d\eta\right) = g(f_\eta H_\xi - f_\xi H_\eta) \quad (8)$$

where $\theta = \theta(H) = p/\rho$ and $H = h/h_e$. The primes in Eqs. (7) and (8) denote differentiation, and the subscripts η and ξ denote partial differentiation with respect to η and ξ, respectively. Ohm's law for the arc may be written

$$I = \frac{2\pi Eg}{pV}\int_0^\infty \sigma\theta d\eta \quad (9)$$

The boundary conditions in the η-ξ plane are

$$H_\eta = f = 0 \text{ at } \eta = \epsilon^*;\ H = f_\eta = 1 \text{ at } \eta = \infty \quad (10)$$

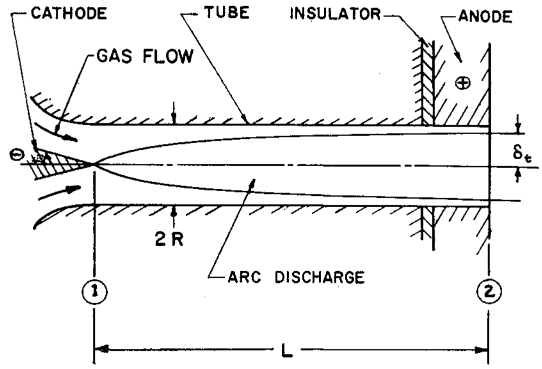

Fig. 1 Tube arc configuration.

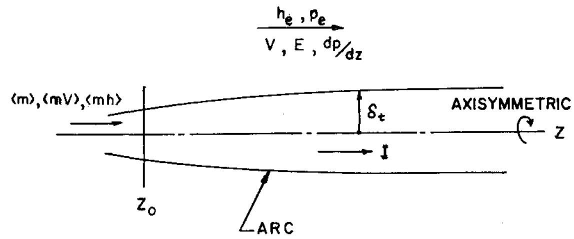

Fig. 2 Diagram for the analytical model of the unconfined arc in forced convection.

where ϵ^* corresponds to ϵ in the r-z plane.

The concept of local similarity[1, 2] provides an approximate method for obtaining a solution of Eqs. (7) and (8). Applying the Karman-Pohlhausen method[3] to Eqs. (7) and (8) yields

$$[d(g\Delta)/d\xi] + (V'/V)(1 + A)(g\Delta) = 0 \quad (11)$$

and

$$\frac{d(g\Lambda)}{d\xi} + \frac{EI}{2\pi h_e} + \lim_{\eta\to\infty}\left(\frac{2kH_\eta}{c_p\theta}\int_0^\eta \theta d\eta\right) = 0 \quad (12)$$

where

$$\Delta = \int_0^\infty f_\eta(1 - f_\eta)d\eta = \text{the momentum thickness} \quad (13)$$

$$\delta = \int_0^\infty\left(\frac{\theta}{\theta_e} - f_\eta\right)d\eta = \text{the displacement thickness} \quad (14)$$

$$\Lambda = \int_0^\infty f_\eta(1 - H)d\eta = \text{the energy thickness} \quad (15)$$

$$A = \delta/\Delta \quad (16)$$

The system of similar equations is derived from Eqs. (7) and (8) by setting the right-hand sides equal to zero. Thus,

$$ff_{\eta\eta} + \beta(\theta/\theta_e - f_\eta^2) = 0 \quad (17)$$

and

$$fH_\eta + \gamma\left[\alpha\sigma\theta + \left(\frac{2kH_\eta}{c_p\theta}\int_0^\eta \theta d\eta\right)_\eta\right] = 0 \quad (18)$$

where

β = a free parameter
$\alpha = E^2\gamma/ph_e$
$\gamma = (\beta/V')^{1/2}$

Equation (10) presents the boundary conditions for the similar equations, that is, Eqs. (17) and (18). Any solution of Eqs. (17) and (18) depends upon the parameters β and E.

Applying the Karman-Pohlhausen integration to Eqs. (17) and (18) yields

$$[1 + \beta(1 + A)]\Delta = 0 \quad (19)$$

and

$$\Lambda = -\gamma\left[\frac{EI}{2\pi h_e} + \lim_{\eta\to\infty}\left(\frac{2kH_\eta}{c_p\theta}\int_0^\eta \theta d\eta\right)\right] \tag{20}$$

The application of local similarity requires that the functions Δ, Λ, and A of Eqs. (11) and (12) be given by the similar solutions [Eqs. (19) and (20)] when an appropriate value of β and the actual values of p, V, V', I, and E are employed. β is an unknown parameter to be determined. Thus, Eqs. (11) and (12) may be written

$$[d(g\Delta)/d\xi] - (V'/V\beta)(g\Delta) = 0 \tag{21}$$

and

$$[d(g\Lambda)/d\xi] - (1/g)(V'/\beta)^{1/2}(g\Lambda) = 0 \tag{22}$$

Equations (21) and (22) together with Ohm's law [Eq. (9)] are three equations in the unknowns β, g, and E. The thicknesses Δ, δ, and Λ depend only upon the unknowns β and E through the similar Eqs. (17) and (18). Whereas it is possible in principle to solve that system of equations for β, g, and E, it may be noted that the equation†

$$\rho_0 v_{z0}\, v_{z0}' = -p' \tag{23}$$

(where the subscript 0 denotes evaluation at $r = \epsilon$) is not necessarily satisfied by the solution.

A function $\eta^*(\xi)$ was introduced, replacing the upper limit ∞ in the statement of the boundary conditions [Eq. (10)] and the upper limits of the integrals in Eqs. (9 and 13–15). $\eta^*(\xi)$ corresponds to δ_t in the r-z coordinate system. The introduction of $\eta^*(\xi)$ permits Eq. (23) to be included in the system of equations to be solved, thereby satisfying the momentum equation near the axis.‡

Making the changes associated with the introduction of η^*, Eqs. (9 and 21–23), with the boundary conditions $<m>$, $<mV>$, and $<mh>$ at $\xi = \xi_0$, may be solved for β, g, E, and η^*, considering the solutions of Eqs. (17) and (18) to be available in terms of β, η^*, and E.

3. Analysis of a Tube Arc

It is convenient to divide the flow field of the tube arc into an inner region (containing the heated gas) and an outer region (see Fig. 1) separated by a boundary δ_t. One may consider that the flow and energy transport processes in the inner region are described by the analysis of the unconfined arc presented in Sec. 2, with the pressure p and the velocity V considered as unknown variables to be determined. The outer flow may be assumed to obey the one-dimensional relations of gasdynamics, and it was assumed that the outer flow is isothermal. The flows in the inner and outer regions were matched by requiring that the conservation equations for mass, momentum, and energy be satisfied for the entire tube.

3.1 Momentum equation for the outer flow

The momentum equation, relating the pressure to the velocity of the outer flow, is

$$\rho_e V V' = -p' \tag{24}$$

where ρ_e is the density of the gas in the outer flow.

3.2 Matching of the inner and outer flows

It may be required that the conservation equations for the inner and outer flows be satisfied separately, or it may be required that the conservation equations for either the inner or the outer flow be satisfied while simultaneously requiring that the corresponding conservation equations be satisfied for the entire tube.

The following equations are useful in connection with the matching conditions:

$$[(\langle mh\rangle - \langle m\rangle h_e)/2\pi h_e] = -g\Lambda \tag{25}$$

$$[(\langle mV\rangle - \langle m\rangle V)/2\pi V] = -g\Delta \tag{26}$$

$$[(\langle m\rangle - \pi R^2\rho_e V)/2\pi] = -g\delta \tag{27}$$

where§

$\langle m\rangle$ = the rate of mass transport through the tube
$\langle mV\rangle$ = the rate of momentum transport through the tube
$\langle mh\rangle$ = $\langle m\rangle\langle h\rangle$ = the rate of energy transport through the tube
$\langle h\rangle$ = the bulk stagnation enthalpy of the gas

Equations (25–27) were derived[2] from equations expressing the rates of mass, momentum, and energy transport through the tube and from the equation

$$\delta_t^2 = \frac{2g}{pV}\int_0^{\eta^*}\theta d\eta \tag{28}$$

which was derived from the first of Eqs. (6), defining the similarity transformation.

3.3 Basic equations for a tube arc

The basic equations describing the arc heating process in a tube arc under the assumptions made herein are summarized below for reference purposes:

1) Inner region: Eqs. (17, 18, and 21–23)
2) Outer region: Eq. (24)
3) Combined flow: Eqs. (25–27)
4) Ohm's law: Eq. (9)
5) Lateral boundary conditions: Eqs. (10) with η^* replacing ∞

3.4 Illustrative example

Let it be required to compute the radial distributions of the properties of the gas and the parameters of the flow, and the current and voltage of the arc of a tube arc operating with a known mass flow rate $<m>$ and a given value of the bulk stagnation enthalpy $<h>$ of the exhaust gases. It is assumed that the operating conditions are such that the thermal boundary does not meet the wall of the tube. The pressure of the reservoir into which the plasma exhausts is specified.

For the example given, the axial boundary conditions are as follows [cf., Eqs. (25–27)]:

$$g\Delta = g\delta = g\Lambda = 0 \text{ at } \xi = 0 \qquad g\Lambda,\ p \text{ are given at } \xi = L \tag{29}$$

Considering the solutions of Eqs. (17) and (18) to be known functions of the unknown parameters g, β, η^*, p, V, and E, Eqs. (9, 21–24, and 27) may be solved simultaneously for those variables after the arc current I is determined. Referring to Eqs. (29), it may be seen that the solution requires solving a two-point boundary value problem. Four boundary conditions are required for the system of Eqs. (9, 21–24, and 27). The fifth condition, given in Eq. (29), is the additional condition necessary to determine the initially unknown constant value of I.

It is clear that a complete solution of the system of equations, including the point by point solution of the similar Eqs. (17) and (18), would require a rather large effort in machine computation. In view of the approximate nature of the analysis, such computations may not be justifiable at present. It is possible to adopt the alternative of choosing

† Derived from Eq. (2) by noting that $v_r = 0$ at $r = \epsilon$.

‡ It is possible to introduce two functions such as η^*, if it is required that Eq. (8), evaluated at the axis, also be satisfied.

§ The items listed denote the rates of transport through the entire tube in the following, whereas they represented the rates of transport through only the arc in the previous discussion.

arbitrary profiles for f and H, each of which involves an arbitrary function of ξ, and to require that the integrated Eqs. (19) and (20) be satisfied along with Eqs. (9, 21–24, and 27). The latter alternative was adopted in Ref. 2, and machine computations are to be made using that approach. Approximate profiles for f and H may be computed from Eqs. (17) and (18), employing the parameters obtained from the foregoing computations.

4. Closure

It is recognized that the substantiation of the results obtained by an analysis such as that discussed in the foregoing must be based on comparisons with experimental data. A complete discussion of the nature of the approximation is beyond the scope of the note presented herein. Additional information may be found in Ref. 2.

References

[1] Hayes, W. D., "On laminar boundary layers with heat transfer," Jet Propulsion **26**, 270–274 (1956).

[2] Skifstad, J. G., "Approximate analysis of an unconfined electric arc in forced convection with application to tube-arc heaters," TM-62-11, Jet Propulsion Center, Purdue Univ. (December 1962).

[3] Pai, S., *Viscous Flow Theory: Laminar Flow* (D. Van Nostrand Co., Inc., New York, 1956), Vol. I, Chap. X.

Temperature Measurement of Hot Gas Streams

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An experimental method is presented with which the temperature of equilibrium, nonradiating, hot gas streams can be measured. The method, consisting of a small probe and companion equations, has been used to measure the temperature of the combustion products of the cyanogen-oxygen flame ($T \approx 7800°R$) at 1 atm pressure; results are within 3% of spectrographic and microwave temperature determinations.

Nomenclature

A	= area, ft^2
C	= specific heat of wall material, Btu/lb-°R
C_p	= specific heat of gas adjacent to probe wall at constant pressure, Btu/lb-°R
D	= diameter of probe, ft
h_e, h_w	= enthalpy at freestream, probe wall, Btu/lb
k	= thermal conductivity of probe wall, Btu/fps-°R
l	= thickness of probe wall, ft
M	= molecular weight
N_{Pr}	= Prandtl number of gas adjacent to wall, $(C_p\mu/k)_w$
P_t, P	= total, static pressure of freestream gas, psia
q	= heat transfer coefficient, Btu/sec-ft^2-°R
r	= nose radius, ft
R	= universal gas constant
T_w, T_0	= temperature of probe wall, initial temperature, °R
u	= velocity of freestream, fps
du/dx	= inviscid velocity gradient at stagnation point, sec^{-1}
x	= distance along probe face, ft
Z	= factor defined by Eq. (4)
γ	= ratio of specific heats $C_p/(C_p - R)$
μ	= viscosity at freestream, lb/ft-sec
ρ, ρ_w	= density of gas at freestream, density of probe wall material, lb/ft^3
τ	= time, sec

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The transient heat transfer technique long has been used by shock-tube and wind-tunnel experimenters to determine the rate of heat transfer from a thermodynamically specifiable test medium to a specific test model. Essentially, the transient technique is a calorimetric method where the heat transfer rate is determined by measuring the time rate of change of the temperature at a point on the model. This technique can be used equally well to measure the rate of heat transfer to a probe from a gas stream whose temperature is to be determined. The main advantage of using the transient technique is that it allows short testing time (of the order of seconds) which, when testing in high-temperature gas streams, eliminates the need for probe cooling.

The temperature of the gas stream is determined uniquely from the heat transfer coefficient in the solution of the Prandtl boundary layer equations. This heat transfer coefficient is defined as the heat transfer rate divided by the difference between the enthalpy of the freestream and that of the gas adjacent to the probe surface. The justification lies in the fact that for simple shapes (axisymmetric stagnation points, flat plate, etc.) the boundary layer equations yield exact solutions within the general boundary layer approximation, even for compressible flow with variable transport properties. Thus, by determining the heat transfer rate experimentally and knowing the enthalpy of the gas adjacent to the probe wall, it is possible to solve directly for the freestream enthalpy and, therefore, the freestream temperature.

The rate at which heat is transferred to a probe, assuming that the probe wall has an infinite thermal conductivity, is

$$dQ/d\tau = \rho_w AlC[\partial(\Delta T)/\partial\tau] \tag{1}$$

At the stagnation point of the probe, Newton's heat transfer equation may be written as

$$dQ/d\tau = qA[h_e + (u^2/2) - h_w] \tag{2}$$

To take into account the actual temperature difference existing across the front face of the probe due to the finite thermal conductivity of the probe wall Z, a correction factor on Eq. (1) is introduced (following Ref. 1):

$$(dQ/d\tau)_{\text{actual}} = Z(dQ/d\tau) \tag{3}$$

For a perfectly insulated wall,

$$Z \approx [1 - (lq/3k)]^{-1} \tag{4}$$

Combining Eqs. (1–3) gives

$$h_e = [Z\rho_w lc/q](\partial T_w/\partial\tau) + h_w - (u^2/2) \tag{5}$$

The stagnation point heat transfer coefficient q, which takes into account real-gas effects for equilibrium air, can be obtained from Ref. 2. However, for an incompressible perfect gas, the heat transfer coefficient has been determined to be[3]

$$q = 0.763(N_{Pr})^{-0.6}(\rho\mu)^{0.5}(du/dx)^{0.5} \tag{6}$$

Since comparison of Eq. (6) with Eq. (79) of Ref. 2 will show that both real-gas effects and compressibility effects have only a small influence when evaluated for velocities below about 21,000 fps, the use of Eq. (6) generally will suffice.

The value of the velocity gradient for a probe having a hemispherical nose in supersonic flow, found from a modified Newtonian flow theory consideration, is

$$du/dx = (1/r)[2(P_t - P)/P_t]^{0.5} \tag{7}$$

The velocity gradient for low subsonic velocities is obtained from potential-flow theory. For a hemispherical nosed probe in incompressible flow,

$$du/dx = 3u/D \tag{8}$$

and, for a flat-faced probe,

$$du/dx = 4u/D \tag{9}$$